

Conformal symmetry and quantum relativity

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The relativistic conception of space and time is challenged by the quantum nature of physical observables. It has been known for a long time that Poincaré symmetry of field theory can be extended to the larger conformal symmetry. We use these symmetries to define quantum observables associated with positions in space-time, in the spirit of Einstein theory of relativity. This conception of localisation may be applied to massive as well as massless fields. Localisation observables are defined as to obey Lorentz covariant commutations relations and in particular include a time observable conjugated to energy. Whilst position components do not commute in presence of a non-vanishing spin, they still satisfy quantum relations which generalise the differential laws of classical relativity. We also give of these observables a representation in terms of canonical spatial positions, canonical spin components and a proper time operator conjugated to mass. These results plead for a new representation not only of space-time localisation but also of motion.

I. INTRODUCTION

Space and time are now considered as closely linked to each other. This is a consequence of relativistic conceptions of space-time which rely on the notion of events localised both in space and time [1]. Time comparison between remote clocks is performed through synchronisation procedures built on the transfer of electromagnetic signals and distance measurement is performed through localisation procedures built on two-way transfer. These ideas are now included in metrological definitions of time and space units as well as in a number of practical applications [2]. It is worth emphasizing that the space and time associated with an event are physical observables delivered by specifically designed apparatus. In particular, the time associated with a given event does not evolve and must therefore be clearly distinguished from any evolution parameter which may be used to write dynamical laws and conservation laws. At the same time, the physical observables describing space-time positions cannot be confused with classical coordinate parameters on a space-time map. Their definition has to reach limits associated with the quantum nature of the physical world [3]. More profoundly, these observables certainly belong to the quantum domain, like atomic clocks used for time definition and electromagnetic signals used for synchronisation or localisation.

The problem of space-time localisation however raises challenging issues in the context of standard quantum formulations. The definition of space positions is delicate in the presence of spin [4–8]. Moreover, time is usually treated as a classical parameter rather than as an operator [9] and the difference in the description of space and time variables leads to considerable difficulties in the attempts to build quantum and relativistic

theories of gravity [10,11]. More generally, the representation of evolution remains a challenge in any quantum framework where the prime roles are played by conserved quantities [12–14]. In the present paper, we show that attaching more importance to the symmetry properties of the physical theory allows to progress towards a solution of these difficulties.

The advent of relativity theory was mainly based upon the symmetry of electromagnetism under Lorentz transformations. Connecting localisation in space-time to propagation of electromagnetic pulses, Einstein was able to derive the relativistic transformations of space and time variables. He then discovered the law of inertia of energy as a consequence of conservation of the generators associated with Lorentz boosts [15]. Shortly after the birth of relativity, Bateman and Cunningham showed that Lorentz symmetry of electromagnetism can be extended to the larger group of conformal coordinate transformations [16] which preserve Maxwell equations while fitting the relativistic definition of uniformly accelerated motion. The conservation of generators associated with this symmetry was soon established [17]. Conformal symmetry implies that the propagation of electromagnetic fields is not sensitive to a conformal variation of the metric tensor, that is a change of space-time scales preserving the velocity of light [18].

The interpretation of conformal symmetry as a general symmetry of the laws of physics has exerted an obvious attraction on a number of physicists but it received at the same time severe objections from other ones and sometimes from the same ones [19–25]. A recurrent matter of debate is the physical significance of the conformal generators. Another one is the pertinence of conformal symmetry for massive fields or massive objects. It has been known for long that mass has to vary under conformal

transformations so that massive field theories have been modified to remain invariant [26,27]. The modification reduces to a transformation of the mass parameter appearing in the equations, which simply means that mass scales vary as reciprocals of space-time scales under conformal transformations. Clearly, this behaviour has to be expected from the dimensional relation which connects mass and space-time as soon as the velocity of light c and the quantum constant \hbar are treated as fundamental constants [27–30]. Despite these attractive features and the lasting efforts of many physicists, among which those of Barut [31], a full recognition of the value of conformal symmetry as an enlarged form of Lorentz symmetry has not already been reached.

The purpose of the present paper is to convince the reader that conformal symmetry is an appropriate tool for addressing the questions raised in the beginning of the Introduction. Conformal symmetry allows to give definitions of space-time observables associated with an event and to show that these definitions fulfill satisfactory quantum and relativistic properties [32–34]. Space-time observables are built on the conformal generators, that is the conserved quantities associated with conformal symmetry. The conformal algebra, that is the commutators of conformal generators, not only fixes the quantum commutators of observables but also their relativistic shifts under changes of reference frames. In particular, the shift of mass under transformations to accelerated frames may be read as defining the space-time observables, in consistency with the redshift of energy and momentum known from classical relativity [35].

When a specific quantum event like the annihilation of an electron-positron pair into a pair of photons is considered, it becomes clear that the space-time position of the annihilation event may be defined, roughly speaking, as the point of coincidence of the two emitted electromagnetic pulses. This definition thus enters the general framework of space-time localisation which has been outlined previously. It nevertheless applies to massive objects such as electron and positron present before the annihilation as well as to massless objects such as the emitted photons, as a direct consequence of conservation laws. The formalism presented in the present paper will allow us to account quantitatively for the various dispersions associated with the quantum nature of the event.

We will then propose further arguments pleading for the physical significance of conformal symmetry. We will use transformation properties of space-time observables to define a conformal factor which plays the role of a metric factor while pertaining to the world of quantum observables. We will also exhibit an equivalent representation of space-time observables in terms of canonical variables.

II. CONFORMAL ALGEBRA

To introduce the algebra of conformal transformations, it is convenient to consider general deformations of a coordinate map which represent changes of frame as Lie transformations

$$x^\mu \rightarrow x^\mu + \delta^\mu(x) \quad (1)$$

where δ^μ are polynomial functions of coordinate parameters x . A commutator between two deformations a and b may be introduced as the difference between the composed deformations $a \circ b$ and $b \circ a$. The difference between the images of a point x through the two deformations is determined by the Lie commutator

$$\delta_{(a,b)}^\mu = \delta_b^\nu \frac{\partial \delta_a^\mu}{\partial x^\nu} - \delta_a^\nu \frac{\partial \delta_b^\mu}{\partial x^\nu} \quad (2)$$

Transformations are performed around an inertial frame, so that tensor indices are raised or lowered by using the Minkowski tensor

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad (3)$$

We consider here the transformations corresponding respectively to translations P_ν , rotations $J_{\nu\rho}$, dilatation D and conformal transformations to uniformly accelerated frames C_ν

$$\begin{aligned} \delta_{P_\nu}^\mu(x) &= \eta_\nu^\mu \\ \delta_{J_{\nu\rho}}^\mu(x) &= \eta_\nu^\mu x_\rho - \eta_\rho^\mu x_\nu \\ \delta_D^\mu(x) &= x^\mu \\ \delta_{C_\nu}^\mu(x) &= 2x_\nu x^\mu - \eta_\nu^\mu x_\rho x^\rho \end{aligned} \quad (4)$$

η_ν^μ denotes a Kronecker symbol.

In quantum field theory, the generators of conformal transformations identify with integrals over a space-like surface of stress tensor components weighted by functions (4) representing map deformations [36]. These generators are quantum observables associated with a given field state and defined in such a manner that they vanish in vacuum. Such a definition is allowed by the conformal invariance of vacuum [37]. The generators are conserved quantities which are used in the following to characterise field states. Their commutators are consistent with Lie commutators (2) and they therefore determine the quantum commutation relations as well as the relativistic shifts of field states under frame transformations [38]. The conformal algebra is described as the set of these commutators

$$\begin{aligned} (P_\mu, P_\nu) &= 0 \\ (J_{\mu\nu}, P_\rho) &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu \\ (J_{\mu\nu}, J_{\rho\sigma}) &= \eta_{\nu\rho} J_{\mu\sigma} + \eta_{\mu\sigma} J_{\nu\rho} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} \\ (D, P_\mu) &= P_\mu \\ (D, J_{\mu\nu}) &= 0 \end{aligned}$$

$$\begin{aligned}
(P_\mu, C_\nu) &= -2\eta_{\mu\nu}D - 2J_{\mu\nu} \\
(J_{\mu\nu}, C_\rho) &= \eta_{\nu\rho}C_\mu - \eta_{\mu\rho}C_\nu \\
(D, C_\mu) &= -C_\mu \\
(C_\mu, C_\nu) &= 0
\end{aligned} \tag{5}$$

The notation (Δ_a, Δ_b) is taken from Lie commutators (2). Quantum commutators $[\Delta_a, \Delta_b]$ are given for any generators Δ_a and Δ_b , and more generally for any observables, by the correspondance rule

$$(\Delta_a, \Delta_b) \equiv \frac{1}{i\hbar} [\Delta_a, \Delta_b] \tag{6}$$

Double commutators obey the Jacobi identity

$$((\Delta_a, \Delta_b), \Delta_c) = (\Delta_a, (\Delta_b, \Delta_c)) - (\Delta_b, (\Delta_a, \Delta_c)) \tag{7}$$

A dot will denote the symmetrised product of operators

$$\Delta_a \cdot \Delta_b \equiv \frac{1}{2} \{\Delta_a \Delta_b + \Delta_b \Delta_a\} \tag{8}$$

III. SPACE-TIME OBSERVABLES

As discussed in the Introduction, localisation of an event in space-time is performed using a field state which, like the field state produced by annihilation of an electron-positron pair, contains photons propagating in at least two different directions. Using the energy-momentum variables P_μ associated with this field state, we define a non vanishing mass M from the Lorentz invariant

$$M = \sqrt{\eta^{\mu\nu} P_\mu P_\nu} = \sqrt{P_\mu P^\mu} \tag{9}$$

Energy-momentum variables P_μ are conserved in the annihilation process and characterize the state of the electron-positron pair existing before the annihilation as well as that of the two-photon state produced by the annihilation. As a consequence, the mass is also conserved in the process. This is also true for the emission-absorption processes considered by Einstein in his derivation of inertia of energy [15].

Space-time observables X_μ , associated with the localisation in space-time of the event, may be built up on the translation, rotation and dilatation generators [33]

$$X_\mu = J_{\nu\mu} \cdot \frac{P^\nu}{M^2} + D \cdot \frac{P_\mu}{M^2} \tag{10}$$

In a simple semi-classical approach, the two photons may be thought as point-like energy distributions propagating along straight lines representing idealised light rays. These two rays are thus considered to intersect each other at the space-time position of the annihilation event. In this context, the observables (10) may be identified with the point of intersection of the two light pulses [33]. This

description is consistent with a classical conception of localisation of events in space-time [1].

Equation (10) provides a more general definition of localisation in space-time which is in particular compatible with the dispersions associated with the quantum nature of the event. Clearly, electron and positron cannot be considered as point-like structures and they bear spin. The emitted photons obey the laws of diffraction and cannot be identified with dimensionless points propagating along idealised light rays. As a consequence, the position of the annihilation event, defined by the coincidence of the two photons, is a fuzzy spot with a dimension of the order of Compton wavelength rather than a sizeless point. The definition (10) leads to a fully quantum description of space-time localisation. In more formal words, such a description involves the enveloping field associated with conformal symmetry, that is the space of rational expressions of conformal generators. The observables X_μ are defined in the enveloping field while forthcoming results are obtained through direct computation in this space.

Using the Jacobi identity (7), we show from conformal algebra (5) that the observables X_μ are shifted under translations, dilatation and rotations exactly as expected from classical relativity

$$\begin{aligned}
(P_\nu, X_\mu) &= -\eta_{\mu\nu} \\
(D, X_\mu) &= -X_\mu \\
(J_{\mu\nu}, X_\rho) &= \eta_{\nu\rho}X_\mu - \eta_{\mu\rho}X_\nu
\end{aligned} \tag{11}$$

The first result means that components of positions obey canonical commutation relations with momenta. In contrast to the situation encountered in standard interpretations of the quantum formalism [9], a time operator has now been defined and energy-time besides momentum-space canonical commutation relations have been obtained. Furthermore, these relations satisfy an explicit Lorentz covariance. This result, as well as the two other ones, convincingly pleads for the interpretation of observables X_μ as positions in space-time associated with a quantum event.

Using the position observables (10), it is possible to write angular momentum components $J_{\mu\nu}$ as sums of external contributions which have their usual form in terms of momenta and positions and of internal observables $S_{\mu\nu}$

$$\begin{aligned}
J_{\mu\nu} &= P_\mu \cdot X_\nu - P_\nu \cdot X_\mu + S_{\mu\nu} \\
S_{\mu\nu} &= \epsilon_{\mu\nu\rho\sigma} S^\rho \frac{P^\sigma}{M} \\
S^\mu &= -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} J_{\nu\rho} \frac{P_\sigma}{M}
\end{aligned} \tag{12}$$

The vector S_μ is the Pauli-Lubanski vector, a covariant generalisation of spin, while $\epsilon_{\mu\nu\lambda\rho}$ is the antisymmetric Lorentz tensor [36]. Spin is transverse with respect to energy-momentum

$$P^\mu S_{\mu\nu} = P_\mu S^\mu = 0 \tag{13}$$

and is invariant under translations and dilatation while being rotated as a vector under rotations

$$\begin{aligned}(P_\mu, S_\rho) &= (D, S_\rho) = 0 \\ (J_{\mu\nu}, S_\rho) &= \eta_{\nu\rho} S_\mu - \eta_{\mu\rho} S_\nu\end{aligned}\quad (14)$$

Spin components obey simple commutation relations

$$(S_\mu, S_\nu) = S_{\mu\nu} \quad (15)$$

The redshift laws for energy-momentum no longer have a classical form in presence of non vanishing spin. Whereas D has a classical form in terms of momenta and positions (10)

$$D = P^\rho \cdot X_\rho \quad (16)$$

this is not the case for angular momentum (12). The redshift of P_ν under the transformation to accelerated frame C_μ thus depends on spin [34] whereas the redshift of mass has a classical form, as a consequence of transversality (13) of spin

$$\begin{aligned}(C_\mu, P_\nu) &= 2(\eta_{\mu\nu} P^\rho \cdot X_\rho - P_\mu \cdot X_\nu + P_\nu \cdot X_\mu - S_{\mu\nu}) \\ (C_\mu, M^2) &= 4M^2 \cdot X_\mu\end{aligned}\quad (17)$$

This mass shift is proportional to mass itself and to the position measured along the direction of acceleration, which is the form expected for the potential energy of a mass in a constant gravitational field [35]. Einstein redshift law is thus recovered for the quantum shift of mass written in terms of quantum space-time observables. This property may in fact be used to define space-time observables [34]. We can also remind that the shifts of observables X_μ under transformations to accelerated frames do not take a classical form, and this is already true for spinless systems [33].

The commutators of different position components provide a further illustration of the changes that have to be performed to shift from a classical to a quantum description of localisation in space-time. It indeed results from conformal algebra (5) that different position components do not commute except for spinless systems

$$(X_\mu, X_\nu) = \frac{S_{\mu\nu}}{M^2} \quad (18)$$

Positions and spin do not commute either

$$(S_\mu, X_\nu) = \frac{P_\mu S_\nu}{M^2} \quad (19)$$

The commutator (18) clearly indicates that the concepts originating from classical relativity have to be modified in a quantum framework. Space-time observables differ from mere coordinate parameters and frame transformations from changes of coordinate maps, since the former are directly related to relativistic symmetries and the latter to conventional parametrizations [39].

The existence of a covariant definition (10) for space-time observables is alluded to in the book of Barut and Raczka [40]. The physical interest of this definition is questioned because covariant commutation relations do

not seem to be compatible with the usual laws of motion. The time component X_0 cannot be identified with the time parameter used in standard quantum formalism and it appears difficult in these conditions to design a canonical Hamiltonian formalism. In the context of the present paper, we may notice a further difficulty raised by the non-commutativity of position components, namely that it might be uneasy to link the shifts of observables with the differential laws typical of classical relativity.

To address these difficulties, we first emphasize once more that the time operator X_0 is a localisation observable, that is precisely the date associated with an event. This observable clearly differs from any kind of evolution parameter in particular because the date associated with an event is a conserved quantity, that is a quantity preserved by evolution [33]. More profoundly, we show in the next sections how to bypass the objections discussed in the previous paragraph. We first derive laws which generalise the differential laws of classical relativity. We then design a canonical representation of localisation observables which is fully equivalent to the covariant representation.

IV. QUANTUM CONFORMAL FACTOR

In classical relativity, each map deformation is associated with a change of the metric tensor. For conformal transformations, this change reduces to a point-dependent rescaling which is deduced from the map deformations (1)

$$\begin{aligned}\frac{\partial \delta_\nu}{\partial x^\mu} + \frac{\partial \delta_\mu}{\partial x^\nu} &= -2\eta_{\mu\nu} \lambda(x) \\ \lambda_{P_\nu}(x) &= 0 \quad \lambda_{J_{\nu\rho}}(x) = 0 \\ \lambda_D(x) &= -1 \quad \lambda_{C_\nu}(x) = -2x_\nu\end{aligned}\quad (20)$$

This is because of this conformal character that these transformations preserve the velocity of light as well as the propagation of massless fields [16]. Notice that the classical expressions (20) are written in terms of classical coordinate parameters and are therefore not properly defined from an operational point of view. This corresponds to the well known fact that metric factors are not relativistic observables [39].

We may however define a conformal factor pertaining to the world of quantum observables. We indeed see by direct inspection that the shift of the mass observable under the action of any conformal generator Δ may be written

$$(\Delta, M) = -M \cdot \lambda(X) \quad (21)$$

where $\lambda(X)$ is the classical function λ evaluated in terms of a quantum argument given by space-time positions X . Mass is invariant under Poincaré transformations while it undergoes the expected space-independent shift under dilatation. The shift of mass under transformations to

accelerated frames is proportional to the position measured along the direction of acceleration, in consistency with Einstein's redshift law.

We show now that this quantum conformal factor may also be defined from variations under translations of the shifts of space-time observables, in conformity with the differential definition (20) of classical relativity. To this purpose, we first write that canonical commutators are classical numbers which commute with any generator Δ

$$(\Delta, (P_\mu, X_\nu)) = 0 \quad (22)$$

In other words, these commutators are invariant under frame transformations or, equivalently, the Planck constant is constant like the velocity of light [29,30], not only under Poincaré transformations but also under dilatation and conformal transformations to accelerated frames. Jacobi identity then entails that space-time shifts (Δ, X_ν) and energy-momentum shifts (Δ, P_μ) are connected through

$$((\Delta, X_\nu), P_\mu) = ((\Delta, P_\mu), X_\nu) \quad (23)$$

Since (Δ, X_ν) is the shift of position, $((\Delta, X_\nu), P_\mu)$ is the variation of this shift under a translation. It is thus a quantum analog of the expression $\frac{\partial \delta_\nu}{\partial x^\mu}$ which appears in the classical equation (20).

It is possible to strengthen this analogy by computing the other double commutator $((\Delta, P_\mu), X_\nu)$ which has the roles of positions and momenta interchanged. For any generator Δ , the momentum shifts (Δ, P_μ) are indeed given by conformal algebra (5) in terms of translations, rotations and dilatation only. For transformations to accelerated frames in particular, the shift is read as

$$(C_\rho, P_\mu) = 2(\eta_{\mu\rho}D + J_{\mu\rho}) \quad (24)$$

We also know that the commutators (11) of translations, rotations and dilatation generators with the observables X_ν have a classical form, so that

$$((C_\rho, P_\mu), X_\nu) = 2(-\eta_{\mu\rho}X_\nu - \eta_{\mu\nu}X_\rho + \eta_{\nu\rho}X_\mu) \quad (25)$$

This is the classical function which already appears in (20) but now evaluated for a quantum argument corresponding to the position observables X . For any generator Δ more generally, the following results are obtained

$$\begin{aligned} ((\Delta, P_\mu), X_\nu) &= -\frac{\partial \delta_\nu}{\partial x^\mu}(X) \\ ((\Delta, X_\nu), P_\mu) &= -\frac{\partial \delta_\nu}{\partial x^\mu}(X) \end{aligned} \quad (26)$$

We have used (23) to deduce the second line from the first one. Quantum analogs of the definition (20) of the conformal factor are now obtained by symmetrising the last expressions in the exchange of the two indices

$$\begin{aligned} ((\Delta, P_\mu), X_\nu) + ((\Delta, P_\nu), X_\mu) &= 2\eta_{\mu\nu}\lambda(X) \\ ((\Delta, X_\nu), P_\mu) + ((\Delta, X_\mu), P_\nu) &= 2\eta_{\mu\nu}\lambda(X) \end{aligned} \quad (27)$$

At this stage, it is worth reminding that the shifts of space-time and momenta observables under transformations to uniformly accelerated frames differ from the rules derived from classical relativity. Meanwhile the commutators with space-time observables cannot be written as differential forms because the position components do not commute in the presence of a non-vanishing spin. Although these expressions differ from their classical analogs, they are nevertheless dictated by conformal algebra. It is in fact a remarkable output of conformal symmetry that the expressions which determine the metric factor may be brought in such a simple manner from the classical to the quantum domain.

Moreover, the relativistic transformations of space-time and energy-momentum scales are now consistently obtained in the quantum domain from the same commutation relations. In the particular case $\mu = \nu = 0$, the first line of (27) is related to variations of redshift of energy-momentum while the second one is related to variations of shifts of space-time observables. Both relations are written in terms of the classical function λ which represents the metric factor in classical relativity. They thus constitute non trivial statements about the relativistic transformation of the quantum observables associated with space, time, energy and momentum. These properties have been derived from conformal symmetry and do not rely on any further assumption, like the ‘‘clock hypothesis’’ of classical relativity [41]. Conformal symmetry is sufficient to force properly defined time observables to have their relativistic transformations determined by the metric factor. To be precise, this behaviour has been demonstrated here for transformations to uniformly accelerated frames. This discussion also means that the conformal factor may be deduced from measurements of field quantities, although propagation of electromagnetic field is known to be insensitive to a conformal variation of the metric tensor.

V. CANONICAL REPRESENTATION

We show in the present section that the covariant formulation of localisation observables that we have discussed up to now may be given a canonical representation.

The problem of representing the position of the center of mass of a system by a quantum operator has been early addressed [4,5]. A momentum representation of localised quantum states has been used to derive differential expressions of spin and position consistent with canonical commutation rules [6]. Operators representing spin and commuting positions with canonical commutators have also been obtained [7]. These definitions, based on purely spatial representations of quantum observables, with time playing the role of an additional parameter,

were designed to develop a Hamiltonian formalism. Although these definitions seem to preclude the possibility of Lorentz invariant commutation relations, we now derive canonical operators from their covariant counterparts, showing that both schemes are equivalent. It will turn out that the canonical formulation not only involves spatial positions and spin components but also a proper time operator conjugate to mass.

The canonical spin may be thought as a representation of spin observables in a center of mass frame [7], that is a frame where momentum has vanishing spatial components. The different spin components can then be rewritten using the transversality property (13)

$$\sigma_j = S_j - \frac{P_j}{P_0 + M} S_0 \quad \sigma^j \sigma_j = S^\mu S_\mu \quad (28)$$

or, conversely,

$$S_0 = -\frac{P^j}{M} \sigma_j \quad S_j = \sigma_j + \frac{P_j}{P_0 + M} S_0 \quad (29)$$

Usual notation σ_j is used for canonical spin variables, with roman characters denoting spatial indices only. Signs have to be manipulated with care since η_{ij} is still used for raising or lowering indices. The canonical spin components commute with energy-momentum

$$(P_\mu, \sigma_j) = (M, \sigma_j) = 0 \quad (30)$$

and their commutation relations may be brought to the usual canonical form

$$(\sigma_i, \sigma_j) = -\epsilon_{ijk} \sigma^k \quad \epsilon_{ijk} \equiv \epsilon_{0ijk} \quad (31)$$

The canonical positions may be understood as the spatial positions of the center of energy of the system. For spinless systems, the position of center of energy is well known to be given by the boost generators, that is the components J_{0j} of angular momentum [15]. For a non vanishing spin, canonical position components have the generalised form

$$\xi_j = \frac{1}{P_0} \cdot J_{0j} - \frac{M S_{0j}}{P_0 (P_0 + M)} \quad (32)$$

Position components have vanishing commutators between themselves as well as with canonical spin components

$$(\xi_i, \xi_j) = 0, \quad (\xi_i, \sigma_j) = 0 \quad (33)$$

Furthermore, they have canonical commutators with momenta while commuting with mass

$$(P_i, \xi_j) = -\eta_{ij}, \quad (M, \xi_j) = 0 \quad (34)$$

The expressions of Poincaré generators in terms of canonical variables constitute quantum generalisations of Einstein's relation between Lorentz boosts and spatial positions [15]

$$\begin{aligned} J_{ij} &= P_i \cdot \xi_j - P_j \cdot \xi_i - \epsilon_{ijk} \sigma^k \\ J_{0i} &= P_0 \cdot \xi_i - \epsilon_{ijk} \frac{\sigma^k P^j}{P_0 + M} \\ P_0 &= \sqrt{M^2 + P_j P_j} \end{aligned} \quad (35)$$

The previous canonical observables have been obtained from the Poincaré generators only. In contrast, the definition (10) of covariant positions requires the presence of an additional generator, namely the dilatation D . This further generator is necessary to obtain the localisation time X_0 . It is also equivalent to the introduction of a further canonical observable besides the canonical spatial positions

$$D = P^j \cdot \xi_j + M \cdot \tau \quad (36)$$

The new time operator τ is seen to commute with all previous canonical observables, i.e. spatial positions ξ_j , momenta P_i and spin components σ_j

$$(\tau, \xi_i) = (\tau, P_i) = (\tau, \sigma_i) = 0 \quad (37)$$

but to be conjugate to the mass observable

$$(M, \tau) = -1 \quad (38)$$

There follows that the covariant positions can be rewritten in terms of equivalent canonical variables including, besides spatial positions and momenta, the mass M and the proper time τ

$$\begin{aligned} X_j &= \xi_j + \frac{P_j}{M} \cdot \tau + \epsilon_{jik} \frac{\sigma^k P^i}{M(P_0 + M)} \\ X_0 &= \frac{P_0}{M} \cdot \tau \end{aligned} \quad (39)$$

Precisely, mass observable M is the translation operator along the direction of momentum while τ is the proper time measured along the same direction.

We have been able to build a canonical formulation equivalent to the covariant one. This formulation includes the definition of a canonical time operator conjugate to the mass observable [42]. It allows us to introduce differential notations for representing commutators with canonical variables

$$\begin{aligned} \frac{\partial F}{\partial \xi^j} &\equiv -(P_j, F) & \frac{\partial F}{\partial P_j} &\equiv (\xi^j, F) \\ \frac{\partial F}{\partial \tau} &\equiv -(M, F) & \frac{\partial F}{\partial M} &\equiv (\tau, F) \end{aligned} \quad (40)$$

It was not possible to define such differential notations for covariant observables because of the ambiguities associated with the non-commutative character of position components. It is important to emphasize that the proper time is, as the other localisation observables, a conserved quantity. Hence, equations involving the symbol $\frac{\partial}{\partial \tau}$ must not be confused with evolution equations.

They rather represent relativistic shifts in frame transformations corresponding to a translation along the direction of momentum. It is worth keeping this precision in mind when comparing the results obtained here with standard Hamiltonian formalism.

As an important application of the canonical representation of observables, we now introduce velocities as derivatives of space-time observables versus proper time

$$V_\mu \equiv \frac{\partial X_\mu}{\partial \tau} = -(M, X_\mu) \quad (41)$$

It follows from the canonical commutator (11) that momenta are related to velocities in a quite simple manner

$$P_\mu = MV_\mu \quad (42)$$

The commutation of mass with momenta may then be read as a quantum version of the law of inertia

$$\frac{\partial P_\mu}{\partial \tau} = M \frac{\partial V_\mu}{\partial \tau} = M \frac{\partial^2 X_\mu}{\partial \tau^2} = 0 \quad (43)$$

This suggests to reconsider the problem of motion in a quantum formalism with the help of the principles of symmetry and, particularly, of conformal symmetry.

VI. CONCLUSION

The localisation of an event in space-time may be characterised by position quantum observables which transform covariantly under Lorentz transformations, and which obey Lorentz covariant commutation relations with momenta. Such position observables have their commutators determined by the spin components. The Lorentz covariant observables may be represented equivalently by canonical operators, at the expense of giving up explicit Lorentz covariance. Canonical observables satisfy usual canonical commutation relations, provided a proper time observable conjugate to the mass observable is included in the canonical representation. The shifts of covariant observables under transformations to accelerated frames are closely connected to each other as well as to the quantum conformal factor.

These results have been obtained as consequences of conformal invariance of massless field theories. In particular, space-time observables have been defined for electromagnetic field states consisting in two photons. As emphasized in the Introduction, these states may be produced by a quantum event such as the annihilation of an $e^+ - e^-$ pair into a pair of photons. Energy-momentum variables are conserved in the process and characterize the $e^+ - e^-$ state as well as the state of the two emitted photons. It is thus clear that the space-time positions defined as the point of coincidence of the two electromagnetic pulses produced by the event, is an image of the space-time position of the annihilation process. This definition thus enters the general framework of space-time

localisation but it may be applied to massive as well as massless objects. As a direct consequence of conservation laws, the conformal variation of mass which has been established for electromagnetic field states has to hold also for massive objects. In particular, the mass of elementary particles such as e^- or e^+ has to vary as the reciprocal of the space-time scale under conformal transformations to accelerated frames.

We are finally led to question not only the representation of localisation in space-time but also the representation of movement inherited from classical physics. To illustrate this point, we consider again the annihilation of an $e^+ - e^-$ pair and we assume that the two particles are bound to each other as a positronium atom moving along a given direction. The annihilation process may occur at different positions in time, each of them corresponding to a different space-time position in the theoretical framework of the present paper. The proper time τ has been defined as a localisation observable, the variation of which describes a translation along the direction of motion. Different values of the proper time variable may therefore be used to distinguish between the different events which may occur along the same trajectory. The commutator of any observable with M may be understood as the derivative of this observable with respect to τ . In particular, a quantum form of the law of inertia is derived in this manner from conformal symmetry.

This leads to a new conception of motion which is now apprehended as a collection of events corresponding to different positions in space-time. Bertrand Russell has lucidly analysed this ‘atomistic’ conception of space-time which is one of the most profound conceptual implications of relativistic theories [43]

The world which the theory of relativity presents to our imagination is not so much a world of ‘things’ in ‘motion’ as a world of *events*.

Such a conception is demanded not only by relativistic arguments discussed by Russell, but also by the necessity of defining well behaved quantum observables. In our opinion, it is quite remarkable that the conformal symmetry, a natural extension of Lorentz symmetry, allows to lay down the foundations of a theoretical framework which has the ability of dealing satisfactorily with relativistic as well as quantum requirements.

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